

# FUNDAMENTAL GROUP OF UNIQUELY ERGODIC CANTOR MINIMAL SYSTEMS

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**ABSTRACT.** We introduce the fundamental group  $\mathcal{F}(\mathcal{R}_{G,\varphi})$  of a uniquely ergodic Cantor minimal  $G$ -system  $\mathcal{R}_{G,\varphi}$  where  $G$  is a countable discrete group. We compute fundamental groups of several uniquely ergodic Cantor minimal  $G$ -systems. We show that if  $\mathcal{R}_{G,\varphi}$  arises from a free action  $\varphi$  of a finitely generated abelian group, then there exists a unital countable subring  $R$  of  $\mathbb{R}$  such that  $\mathcal{F}(\mathcal{R}_{G,\varphi}) = R_+^\times$ . We also consider the relation between fundamental groups of uniquely ergodic Cantor minimal  $\mathbb{Z}^n$ -systems and fundamental groups of crossed product  $C^*$ -algebras  $C(X) \rtimes_\varphi \mathbb{Z}^n$ .

## 1. INTRODUCTION

Let  $M$  be a factor of type  $\text{II}_1$  with a normalized trace  $\tau$ . Murray and von Neumann introduced the fundamental group  $\mathcal{F}(M)$  of  $M$  in [32]. The fundamental group of  $M$  is defined as the set of the numbers  $\tau \otimes \text{Tr}(p)$  for some projection  $p \in M_n(M)$  such that  $pM_n(M)p$  is isomorphic to  $M$  where  $\text{Tr}$  is the usual unnormalized trace on  $M_n(\mathbb{C})$ . They showed that if  $M$  is hyperfinite, then  $\mathcal{F}(M) = \mathbb{R}_+^\times$ . Since then there has been many works on the computation of fundamental groups. Voiculescu [42] showed that  $\mathcal{F}(L(\mathbb{F}_\infty))$  of the group factor of the free group  $\mathbb{F}_\infty$  contains the positive rationals and Radulescu proved that  $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+^\times$  in [40]. Connes [6] showed that if  $G$  is an ICC group with property (T), then  $\mathcal{F}(L(G))$  is a countable group. Popa showed that any countable subgroup of  $\mathbb{R}_+^\times$  can be realized as the fundamental group of some factor of type  $\text{II}_1$  in [36]. Furthermore Popa and Vaes [38] exhibited a large family  $\mathcal{S}$  of subgroups of  $\mathbb{R}_+^\times$ , containing  $\mathbb{R}_+^\times$  itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension in  $(0, 1)$ , such that for each  $H \in \mathcal{S}$  there exist many free ergodic measure preserving actions of  $\mathbb{F}_\infty$  for which the associated  $\text{II}_1$  factor  $M$  has the fundamental group equal to  $H$ .

Popa's results are based on the study of rigidity properties of  $\text{II}_1$  factors  $L^\infty(X) \rtimes_T G$  and orbit equivalence relations  $\mathcal{R}_{G,T}$  arising from (free) ergodic measure preserving actions  $T$  of countable groups  $G$  on probability measure spaces  $(X, \mu)$  via the group measure space construction in [31]. (See also [10] and [11] for its generalization.) The fundamental group of  $\mathcal{R}_{G,T}$  can also be defined as the set of numbers  $\mu \times \delta(Y)$  for some measurable set  $Y$  in  $X \times \{1, \dots, n\}$  such that  $\mathcal{R}_{G,T}^n|_Y$  is orbit equivalent to  $\mathcal{R}_{G,T}$  where  $\delta$  is the counting measure. It is easy to see that  $\mathcal{F}(\mathcal{R}_{G,T}) \subseteq \mathcal{F}(L^\infty(X) \rtimes_T G)$ . This inclusion may be strict. (See 6.1 in [37].)

The result in [7] implies  $\mathcal{F}(\mathcal{R}_{G,T}) = \mathbb{R}_+^\times$  whenever  $G$  is amenable. Gelfert and Golodets [14] showed that there exist ergodic measure preserving actions having trivial fundamental group before Popa's result [35]. Moreover Gaboriau [13] showed that if  $\mathcal{R}_{\mathbb{F}_n,T}$  is an orbit equivalence relation arising from a free ergodic measure preserving action  $T$  of a non-amenable free group of finite rank, then  $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n,T}) = \{1\}$ . On the contrary, Popa and Vaes showed that for each  $H \in \mathcal{S}$  there exist

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many orbit equivalence relations  $\mathcal{R}_{\mathbb{F}_\infty, T}$  arising from free ergodic measure preserving actions  $T$  of  $\mathbb{F}_\infty$  such that  $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty, T}) = H$ . For a countable discrete group  $G$ , it is an interesting problem to consider how  $G$  affect  $\mathcal{F}(\mathcal{R}_{G, T})$ .

Watatani and the author introduced the fundamental group  $\mathcal{F}(A)$  of a simple unital  $C^*$ -algebra  $A$  with a normalized trace  $\tau$  based on the computation of Picard groups by Kodaka [23], [24] and [25]. The fundamental group  $\mathcal{F}(A)$  is defined as the set of the numbers  $\tau \otimes \text{Tr}(p)$  for some projection  $p \in M_n(A)$  such that  $pM_n(A)p$  is isomorphic to  $A$ . We computed the fundamental groups of several  $C^*$ -algebras and showed that any countable subgroup of  $\mathbb{R}_+^\times$  can be realized as the fundamental group of a separable simple unital  $C^*$ -algebra with a unique trace in [33] and [34]. Note that the fundamental groups of separable simple unital  $C^*$ -algebras are countable.

We consider topological dynamical systems on the Cantor set in this paper. Giordano, Putnam and Skau classified Cantor minimal  $\mathbb{Z}$ -systems up to (topological) orbit equivalence [17]. They also introduced the notion of strong orbit equivalence for Cantor minimal  $\mathbb{Z}$ -system and showed that two Cantor minimal  $\mathbb{Z}$ -system are strong orbit equivalent if and only if their associated  $C^*$ -algebras are isomorphic. Giordano, Matui, Putnam and Skau extends the classification up to orbit equivalence of Cantor minimal systems arising from actions of finitely generated abelian groups [15] and [16].

In this paper we introduce the fundamental group of a uniquely ergodic Cantor minimal  $G$ -system  $\mathcal{R}_{G, \varphi}$  where  $G$  is a countable discrete group. The fundamental group  $\mathcal{F}(\mathcal{R}_{G, \varphi})$  of a uniquely ergodic Cantor minimal  $G$ -system  $\mathcal{R}_{G, \varphi}$  is defined as the set of numbers  $\mu \times \delta(U)$  for some clopen set  $U$  in  $X \times \{1, \dots, n\}$  such that  $\mathcal{R}_{G, \varphi}^n|_U$  is (topologically) orbit equivalent to  $\mathcal{R}_{G, \varphi}$  where  $\mu$  is a unique invariant probability measure and  $\delta$  is the counting measure. We show that  $\mathcal{F}(\mathcal{R}_{G, \varphi})$  is a countable multiplicative subgroup of  $\mathbb{R}_+^\times$ . Note that we cannot show this theorem in the same way as in the case of ergodic measure preserving actions of countable groups on probability measure spaces because we do not know whether the analogous result of Hopf equivalence theorem (see, for example, Proposition 3.3 in [10]) is true for Cantor minimal  $G$ -systems. Moreover we do not know whether  $\mu \times \delta(U_1) = \mu \times \delta(U_2)$  implies that  $\mathcal{R}_{G, \varphi}^n|_{U_1}$  is orbit equivalent to  $\mathcal{R}_{G, \varphi}^n|_{U_2}$  in general.

We show that for any unital countable subring  $R$  of  $\mathbb{R}$ , there exists a Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi}) = R_+^\times$ . Conversely, we also show that if  $\mathcal{R}_{G, \varphi}$  arises from a free action of a finitely generated abelian group, then there exists a unital countable subring  $R$  of  $\mathbb{R}$  such that  $\mathcal{F}(\mathcal{R}_{G, \varphi}) = R_+^\times$ . Therefore  $\{9^n : n \in \mathbb{Z}\}$  cannot be realized as the fundamental group of a Cantor minimal system in this class.

We do not know whether there exists a relation between orbit equivalence of Cantor minimal  $G$ -systems and  $C^*$ -isomorphism of associated  $C^*$ -algebras in general. We show that if  $\mathcal{R}_{\mathbb{Z}^n, \varphi}$  arises from a free minimal action  $\varphi$  of  $\mathbb{Z}^n$  on a Cantor set  $X$ , then  $\mathcal{F}(C(X) \rtimes_\varphi \mathbb{Z}^n) \subseteq \mathcal{F}(\mathcal{R}_{\mathbb{Z}^n, \varphi})$ . Note that we do not know whether  $C(X) \rtimes_\varphi \mathbb{Z}^2$  belongs to classifiable classes by  $K$ -groups. We also show that there exists a Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that  $\mathcal{F}(C(X) \rtimes_\varphi \mathbb{Z}) \neq \mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi})$ .

In section 5 we compute fundamental groups of several uniquely ergodic Cantor minimal systems. Some computations suggest that the fundamental groups of Cantor minimal systems have arithmetical flavor compared with the case of ergodic measure preserving actions on probability measure spaces.

## 2. CANTOR MINIMAL SYSTEMS

In this section we review some definitions and results on Cantor minimal systems. See, for example, [16] and [18] for details. Let  $X$  be the Cantor set, that is,  $X$  is a compact separable totally disconnected metric space without isolated points. For

a free action  $\varphi$  of a countable discrete group  $G$  on  $X$  by homeomorphisms (we call this dynamical system the *Cantor  $G$ -system*), an orbit equivalence relation  $\mathcal{R}_{G,\varphi}$  is defined by

$$\mathcal{R}_{G,\varphi} = \{(x, \varphi_g(x)) \in X \times X : x \in X, g \in G\}.$$

For two orbit equivalence relations  $\mathcal{R}_1$  on  $X_1$  and  $\mathcal{R}_2$  on  $X_2$ , we say that  $\mathcal{R}_1$  is (*topologically*) *orbit equivalent* to  $\mathcal{R}_2$  if there exists a homeomorphism  $F$  of  $X_1$  onto  $X_2$  such that  $F \times F(\mathcal{R}_1) = \mathcal{R}_2$ . We denote by  $\text{Aut}(\mathcal{R})$  the set of homeomorphism of  $X$  such that  $F \times F(\mathcal{R}) = \mathcal{R}$ . An orbit equivalence relation  $\mathcal{R}$  on  $X$  is said to be *minimal* if  $\mathcal{R}[x] = \{y \in X : (x, y) \in \mathcal{R}\}$  is dense in  $X$  for any  $x \in X$ . An  $\mathcal{R}$ -*invariant measure* is a Borel measure  $\mu$  on  $X$  such that  $\mu$  is  $h$ -invariant for any  $h \in \text{Homeo}(X)$  satisfying  $(x, h(x)) \in \mathcal{R}$  for all  $x \in X$ . Let  $M(\mathcal{R})$  denote the set of  $\mathcal{R}$ -invariant probability measures. An orbit equivalence relation  $\mathcal{R}$  is said to be *uniquely ergodic* if  $M(\mathcal{R})$  has exactly one element. These definitions coincide with usual definitions for topological dynamical systems.

Assume that  $M(\mathcal{R}) \neq \emptyset$ . Let  $D_m(X, \mathcal{R})$  be a quotient group of  $C(X, \mathbb{Z})$  by  $\{f \in C(X, \mathbb{Z}) : \int_X f d\mu = 0 \text{ for all } \mu \in M(\mathcal{R})\}$ . This countable abelian group  $D_m(\mathcal{R})$  has an order structure such that the positive cone  $D_m(\mathcal{R})^+$  is the set of all  $[f]$ , where  $f \geq 0$ . For a clopen set  $U$  of  $X$ , we denote by  $1_U$  the characteristic function on  $U$ , that is,  $1_U(x) = 1$  if  $x \in U$  and  $1_U(x) = 0$  if  $x \notin U$ . We say that two triples  $(D_m(\mathcal{R}_1), D_m(\mathcal{R}_1)^+, [1]_{X_1})$  and  $(D_m(\mathcal{R}_2), D_m(\mathcal{R}_2)^+, [1]_{X_2})$  are *isomorphic* if there exists an order isomorphism  $\psi$  of  $D_m(\mathcal{R}_1)$  onto  $D_m(\mathcal{R}_2)$  such that  $\psi([1]_{X_1}) = [1]_{X_2}$ . A triple  $(D_m(\mathcal{R}), D_m(\mathcal{R})^+, [1]_X)$  is an orbit equivalence invariant for Cantor minimal systems. Giordano, Putnam, Matui and Skau actually showed that this triple is a complete invariant for AF relations and the case where  $G$  is a finitely generated abelian group. (See Theorem 2.5 in [16].)

**Remark 2.1.** If  $\mathcal{R}$  is uniquely ergodic with an  $\mathcal{R}$ -invariant probability measure  $\mu$ , then the map  $T_\mu$  of  $D_m(\mathcal{R})$  into  $\mathbb{R}$  defined by  $T_\mu([f]) = \int_X f d\mu$  is an order isomorphism of  $D_m(\mathcal{R})$  onto  $T_\mu(D_m(\mathcal{R})) \subseteq \mathbb{R}$  with  $T_\mu([1]_X) = 1$ . Moreover  $T_\mu(D_m(\mathcal{R}))$  is an additive subgroup of  $\mathbb{R}$  generated by  $\{\mu(U) : U \text{ is a clopen set in } X\}$  because for any  $f \in C(X, \mathbb{Z})$ , there exist clopen sets  $U_1, \dots, U_n$  in  $X$  and integers  $m_1, \dots, m_n$  such that  $f = \sum_{k=1}^n m_k 1_{U_k}$ .

For a Cantor system  $\mathcal{R}_{G,\varphi}$  and a natural number  $n$ , define an orbit equivalence relation  $\mathcal{R}_{G,\varphi}^n$  on  $X \times \{1, \dots, n\}$  by  $\{(x, i), (y, j) : (x, y) \in \mathcal{R}_{G,\varphi}, i, j \in \{1, \dots, n\}\}$ . Note that  $X \times \{1, \dots, n\}$  is a Cantor set and  $\mathcal{R}_{G,\varphi}^n$  arises from an action  $\tilde{\varphi}$  of  $G \times \mathbb{Z}/n\mathbb{Z}$  on  $X \times \mathbb{Z}/n\mathbb{Z}$ . If  $\mathcal{R}_{G,\varphi}$  is a Cantor minimal system, then  $\mathcal{R}_{G,\varphi}^n$  is minimal. It can be easily checked that for any  $\mathcal{R}_{G,\varphi}^n$ -invariant probability measure  $\nu$  on  $X \times \{1, \dots, n\}$ , there exists an  $\mathcal{R}_{G,\varphi}$ -invariant probability measure  $\mu$  on  $X$  such that  $\nu = \mu \times \frac{1}{n} \delta$  where  $\delta$  is the counting measure on  $\{1, \dots, n\}$ .

**Proposition 2.2.** Let  $\mathcal{R}_{G,\varphi}$  be a Cantor system and  $n$  a natural number. Then

$$(D_m(\mathcal{R}_{G,\varphi}^n), D_m(\mathcal{R}_{G,\varphi}^n)^+, [1]_{X \times \{1, \dots, n\}}) \cong (D_m(\mathcal{R}_{G,\varphi}), D_m(\mathcal{R}_{G,\varphi})^+, n[1]_X).$$

*Proof.* Let  $\psi$  be an order homomorphism of  $D_m(\mathcal{R}_{G,\varphi})$  to  $D_m(\mathcal{R}_{G,\varphi}^n)$  such that  $\psi([1]_V) = [1]_{V \times \{1\}}$  for any clopen set  $V$  in  $X$ . It is easy to see that  $\psi$  is well-defined and injective. For any clopen set  $U$  in  $X \times \{1, \dots, n\}$ , there exist clopen sets  $V_1, \dots, V_n$  in  $X$  such that  $U = \cup_{k=1}^n V_k \times \{k\}$ . Since  $[1]_{V_k \times \{k\}} = [1]_{V_k \times \{1\}}$  in  $D_m(\mathcal{R}_{G,\varphi}^n)$  for any  $1 \leq k \leq n$ , we see that  $\psi$  is surjective and  $n\psi([1]_X) = [1]_{X \times \{1, \dots, n\}}$ . Therefore we obtain the conclusion.  $\square$

Let  $U$  be a clopen set in  $X$ . Define an orbit equivalence relation  $\mathcal{R}_{G,\varphi}|_U$  by  $\mathcal{R}_{G,\varphi} \cap U \times U$ . If  $\mathcal{R}_{\mathbb{Z},\varphi}$  is a Cantor minimal  $\mathbb{Z}$ -system, then  $\mathcal{R}_{\mathbb{Z},\varphi}|_U$  is equal to the induced system  $\mathcal{R}_{\mathbb{Z},\varphi|_U}$  in [17]. It is easy to see that if  $\mathcal{R}_{G,\varphi}$  is minimal, then  $\mathcal{R}_{G,\varphi}|_U$  is minimal.

**Proposition 2.3.** Let  $\mathcal{R}_{G,\varphi}$  be a Cantor minimal system and  $U$  a clopen set in  $X$ . Then there exists a bijective map  $\pi$  of  $M(\mathcal{R}_{G,\varphi})$  onto  $M(\mathcal{R}_{G,\varphi}|_U)$  such that  $\pi(\mu) = \mu|_U$ . Moreover we have

$$(D_m(\mathcal{R}_{G,\varphi}|_U), D_m(\mathcal{R}_{G,\varphi}|_U)^+, [1_U]) \cong (D_m(\mathcal{R}_{G,\varphi}), D_m(\mathcal{R}_{G,\varphi})^+, [1_U]).$$

*Proof.* It is easy to see that  $\pi$  is well-defined. Since  $X$  is compact and  $\mathcal{R}_{G,\varphi}$  is minimal, there exist  $g_1, \dots, g_n \in G$  such that  $X = \cup_{k=1}^n \varphi_{g_k}(U)$ . Put  $U_k := \varphi_{g_k}(U) \setminus (\cup_{i=1}^{k-1} \varphi_{g_i}(U))$  for  $1 \leq k \leq n$ . Then we have  $\mu(V) = \sum_{k=1}^n \mu(\varphi_{g_k}^{-1}(V \cap U_k))$  for any  $\mu \in M(\mathcal{R}_{G,\varphi})$  and any clopen set  $V$  in  $X$ . Since  $\varphi_{g_k}^{-1}(V \cap U_k)$  is a clopen set in  $U$ ,  $\mu$  is determined by  $\mu|_U$ . Therefore  $\pi$  is injective. For any  $\nu \in M(\mathcal{R}_{G,\varphi}|_U)$ , define a measure  $\mu$  on  $X$  by  $\mu(V) = \sum_{k=1}^n \nu(\varphi_{g_k}^{-1}(V \cap U_k))$  for any clopen set  $V$  in  $X$ . We shall show that  $\mu$  is  $\mathcal{R}_{G,\varphi}$ -invariant. Note that for  $1 \leq k, j \leq n$ ,  $g \in G$  and a clopen set  $V$  in  $X$ , there exist homeomorphisms  $h_{k,j,g}$  of  $U$  such that  $h_{k,j,g}(x) = x$  if  $x \notin \varphi_{g_k}^{-1}(U_k \cap \varphi_g(U_j \cap V))$  and  $h_{k,j,g}(x) = \varphi_{g_j}^{-1} \circ \varphi_g^{-1} \circ \varphi_{g_k}(x)$  if  $x \in \varphi_{g_k}^{-1}(U_k \cap \varphi_g(U_j \cap V))$ . It is clear that  $(x, h_{k,j,g}(x)) \in \mathcal{R}_{G,\varphi}|_U$  for any  $x \in U$ . For any  $g \in G$  and any clopen set  $V$  in  $X$ , we have

$$\begin{aligned} \mu(\varphi_g(V)) &= \sum_{k=1}^n \nu(\varphi_{g_k}^{-1}(U_k \cap \varphi_g(V))) \\ &= \sum_{k=1}^n \nu(\varphi_{g_k}^{-1}(U_k \cap \varphi_g(\cup_{j=1}^n U_j \cap V))) \\ &= \sum_{k=1}^n \sum_{j=1}^n \nu(\varphi_{g_k}^{-1}(U_k \cap \varphi_g(U_j \cap V))) \\ &= \sum_{k=1}^n \sum_{j=1}^n \nu(h_{k,j,g}(\varphi_{g_k}^{-1}(U_k \cap \varphi_g(U_j \cap V)))) \\ &= \sum_{k=1}^n \sum_{j=1}^n \nu(\varphi_{g_j}^{-1}(\varphi_{g^{-1}}(U_k) \cap U_j \cap V)) \\ &= \sum_{j=1}^n \nu(\varphi_{g_j}^{-1}(\varphi_{g^{-1}}(\cup_{k=1}^n U_k) \cap U_j \cap V)) \\ &= \mu(V). \end{aligned}$$

Therefore we see that  $\pi$  is surjective.

Define a homomorphism  $\psi$  of  $D_m(\mathcal{R}_{G,\varphi})|_U$  to  $D_m(\mathcal{R}_{G,\varphi})$  by  $\psi([1_W]) = [1_W]$  for any clopen set  $W$  in  $U$ . Then it can be easily checked that  $\psi$  is an order isomorphism by a similar argument.  $\square$

### 3. ANALOGOUS RESULT OF BROWN'S LEMMA

In this section we shall show the analogous result of Lemma 2.5 in [4]. Let  $\mathcal{R}_{G,\varphi}$  be a Cantor minimal system. Define an orbit equivalence relation  $\mathcal{R}_{G,\varphi}^\infty$  on  $X \times \mathbb{N}$  by  $\{((x, i), (y, j)) : (x, y) \in \mathcal{R}_{G,\varphi}, i, j \in \mathbb{N}\}$ . Note that  $X \times \mathbb{N}$  is a locally compact separable totally disconnected metric space without isolated points and  $\mathcal{R}_{G,\varphi}^\infty$  arises from an action  $\tilde{\varphi}$  of  $G \times \mathbb{Z}$  on  $X \times \mathbb{Z}$ . It can be easily checked that if  $\mathcal{R}_{G,\varphi}$  is a uniquely ergodic Cantor minimal system, then  $\mathcal{R}_{G,\varphi}^\infty$  is minimal and has unique (up to scalar multiple)  $\mathcal{R}_{G,\varphi}^\infty$ -invariant measure.

**Lemma 3.1.** Let  $\mathcal{R}_{G,\varphi}$  be a Cantor minimal system on  $X$  and  $U$  a clopen set in  $X$ . Then there exists an injective continuous open map  $F$  of  $X \times \mathbb{N}$  to  $U \times \mathbb{N}$  such that  $(y_1, y_2) \in \mathcal{R}_{G,\varphi}^\infty$  if and only if  $(F(y_1), F(y_2)) \in \mathcal{R}_{G,\varphi}^\infty$ .

*Proof.* Since  $X$  is compact and  $\mathcal{R}_{G,\varphi}$  is minimal, there exist  $g_1, \dots, g_n \in G$  such that  $X = \bigcup_{k=1}^n \varphi_{g_k}(U)$ . Put  $U_k := \varphi_{g_k}(U) \setminus (\bigcup_{i=1}^{k-1} \varphi_{g_i}(U))$  for  $1 \leq k \leq n$ . Let  $\Psi$  be a bijective map of  $\{1, \dots, n\} \times \mathbb{N}$  onto  $\mathbb{N}$ . Define a map  $F$  of  $X \times \mathbb{N}$  to  $U \times \mathbb{N}$  by  $F((x, i)) = (\varphi_{g_k}^{-1}(x), \Psi(k, i))$  if  $x \in U_k$  for any  $1 \leq k \leq n$ . Then it can be easily checked that  $F$  is an injective continuous open map such that  $(y_1, y_2) \in \mathcal{R}_{G,\varphi}^\infty$  if and only if  $(F(y_1), F(y_2)) \in \mathcal{R}_{G,\varphi}^\infty$ .  $\square$

The following lemma is the analogous result of Lemma 2.5 in [4].

**Lemma 3.2.** Let  $\mathcal{R}_{G,\varphi}$  be a Cantor minimal system on  $X$  and  $U$  a clopen set in  $X$ . Then there exists a homeomorphism  $\Phi$  of  $U \times \mathbb{N}$  onto  $X \times \mathbb{N}$  such that  $\Phi \times \Phi((\mathcal{R}_{G,\varphi}|_U)^\infty) = \mathcal{R}_{G,\varphi}^\infty$  and  $\Phi(U \times \{1\}) = U \times \{1\} \subseteq X \times \{1\}$ .

*Proof.* We shall construct a homeomorphism  $\Phi$  of  $U \times \mathbb{N} \times \mathbb{N}$  onto  $X \times \mathbb{N} \times \mathbb{N}$ . First, for  $(x, i, 1) \in U \times \mathbb{N} \times \mathbb{N}$ , define  $\Phi((x, i, 1)) = (x, i, 1) \in X \times \mathbb{N} \times \mathbb{N}$ . Let  $F$  be an injective continuous open map of  $X \times \mathbb{N}$  to  $U \times \mathbb{N}$  in Lemma 3.1. Put  $E_1 = F((X \times \mathbb{N}) \setminus (U \times \mathbb{N}))$ . Define  $\Phi((x, i, 2)) = (F^{-1}((x, i)), 1)$  if  $(x, i) \in E_1$  and  $\Phi((x, i, 2)) = (x, i, 2)$  if  $(x, i) \in (U \times \mathbb{N}) \setminus E_1$ . Put  $E_2 := F((X \times \mathbb{N}) \setminus ((U \times \mathbb{N}) \setminus E_1))$ . In a similar way, define  $\Phi((x, i, 3)) = (F^{-1}((x, i)), 2)$  if  $(x, i) \in E_2$  and  $\Phi((x, i, 3)) = (x, i, 3)$  if  $(x, i) \in (U \times \mathbb{N}) \setminus E_2$ . We construct inductively  $\Phi$  as follows: Let  $E_n := F((X \times \mathbb{N}) \setminus ((U \times \mathbb{N}) \setminus E_{n-1}))$ , and define  $\Phi((x, i, n+1)) = (F^{-1}((x, i)), n)$  if  $(x, i) \in E_n$  and  $\Phi((x, i, n+1)) = (x, i, n+1)$  if  $(x, i) \in (U \times \mathbb{N}) \setminus E_n$ . By a Cantor-Bernstein type argument, we obtain the conclusion.  $\square$

#### 4. FUNDAMENTAL GROUP

Let  $\mathcal{R}_{G,\varphi}$  be a uniquely ergodic Cantor minimal system with an  $\mathcal{R}_{G,\varphi}$ -invariant probability measure  $\mu$ . Put

$$\mathcal{F}(\mathcal{R}_{G,\varphi}) := \{\mu \times \delta(U) : U \text{ is a clopen set in } X \times [n] \text{ such that } \mathcal{R}_{G,\varphi} \sim_{OE} \mathcal{R}_{G,\varphi}^n|_U\}$$

where  $\delta$  is the counting measure on  $[n] = \{1, \dots, n\}$ .

**Theorem 4.1.** Let  $\mathcal{R}_{G,\varphi}$  be a uniquely ergodic Cantor minimal system with an  $\mathcal{R}_{G,\varphi}$ -invariant probability measure  $\mu$ . Then  $\mathcal{F}(\mathcal{R}_{G,\varphi})$  is a countable multiplicative subgroup of  $\mathbb{R}_+^\times$ .

*Proof.* Put

$$\mathfrak{S}(\mathcal{R}_{G,\varphi}) = \{\lambda : \mu \times \delta \circ F = \lambda \mu \times \delta, F \in \text{Aut}(\mathcal{R}_{G,\varphi}^\infty)\}.$$

It is easy to see that  $\mathfrak{S}(\mathcal{R}_{G,\varphi})$  is a multiplicative subgroup of  $\mathbb{R}_+^\times$ . We shall show that  $\mathcal{F}(\mathcal{R}_{G,\varphi}) = \mathfrak{S}(\mathcal{R}_{G,\varphi})$ .

Let  $\lambda \in \mathfrak{S}(\mathcal{R}_{G,\varphi})$ , then there exists a homeomorphism  $F$  of  $X \times \mathbb{N}$  such that  $\mu \times \delta \circ F = \lambda \mu \times \delta$  and  $F \times F(\mathcal{R}_{G,\varphi}^\infty) = \mathcal{R}_{G,\varphi}^\infty$ . We see that  $\mathcal{R}_{G,\varphi}|_{X \times \{1\}}$  is orbit equivalent to  $\mathcal{R}_{G,\varphi}^\infty|_{F(X \times \{1\})}$ . Note that  $\mathcal{R}_{G,\varphi}^\infty|_{X \times \{1\}}$  is orbit equivalent to  $\mathcal{R}_{G,\varphi}$ . Since  $F(X \times \{1\})$  is compact and  $F(X \times \{1\}) \subseteq \bigcup_{i=1}^\infty \{(x, k) \in X \times \mathbb{N} : k = i\}$ , there exist a natural number  $n$  and a clopen set  $U$  in  $X \times \{1, \dots, n\}$  such that  $F(X \times \{1\}) = U$ . Therefore  $\lambda \in \mathcal{F}(\mathcal{R}_{G,\varphi})$ .

Conversely, let  $\lambda \in \mathcal{F}(\mathcal{R}_{G,\varphi})$ . There exist a natural number  $n$ , a clopen set  $U$  in  $X \times \{1, \dots, n\}$  and a homeomorphism  $h$  of  $X$  onto  $U \times \{1, \dots, n\}$  such that  $h \times h(\mathcal{R}_{G,\varphi}) = \mathcal{R}_{G,\varphi}^n|_U$ . By Lemma 3.2, there exist a homeomorphism  $\Phi$  of  $U \times \mathbb{N}$  onto  $X \times \{1, \dots, n\} \times \mathbb{N}$ . Let  $\Psi$  be a bijective map of  $\{1, \dots, n\} \times \mathbb{N}$  onto  $\mathbb{N}$ , and define a homeomorphism  $F$  of  $X \times \mathbb{N}$  by  $F = (id \times \Psi) \circ \Phi \circ (h \times id)$ . It can be easily checked  $F \in \text{Aut}(\mathcal{R}_{G,\varphi}^\infty)$  and  $\mu \times \delta(F(X \times \{1\})) = \mu \times \delta(U) = \lambda$ . Therefore  $\lambda \in \mathfrak{S}(\mathcal{R}_{G,\varphi})$ .

It is clear that  $\mathcal{F}(\mathcal{R}_{G,\varphi})$  is countable because  $X$  is the Cantor set.  $\square$

**Definition 4.2.** Let  $\mathcal{R}_{G,\varphi}$  be a uniquely ergodic Cantor minimal system with an  $\mathcal{R}_{G,\varphi}$ -invariant probability measure  $\mu$ . We call  $\mathcal{F}(\mathcal{R}_{G,\varphi})$  the fundamental group of  $\mathcal{R}_{G,\varphi}$ , which is a multiplicative (countable) subgroup of  $\mathbb{R}_+^\times$ .

**Remark 4.3.** (i) We need not assume that  $\varphi$  is free. Moreover we can define the fundamental group of étale equivalence relations on Cantor sets by Proposition 2.3 in [18]. But the associated  $C^*$ -algebra  $C(X) \rtimes_\varphi G$  ( $C_r^*(\mathcal{R})$ ) may not be simple. In this paper, we assume that  $\varphi$  is free.

(ii) We do not know  $\mu \times \delta(U_1) = \mu \times \delta(U_2)$  implies that  $\mathcal{R}_{G,\varphi}^n|_{U_1}$  is orbit equivalent to  $\mathcal{R}_{G,\varphi}^n|_{U_2}$  in general. It is known that if  $\mathcal{R}_{G,\varphi}$  is a Cantor minimal  $\mathbb{Z}^m$ -system, then  $\mathcal{R}_{G,\varphi}$  satisfies the analogous result of Hopf equivalence theorem. See, for example, [26] (Theorem 3.20) and [29] (Theorem 6.11).

(iii) Since  $M(\mathcal{R}_{G,\varphi}) = \{\mu\}$ , we may regard  $\mathcal{R}_{G,\varphi}$  as an orbit equivalence relation  $\mathcal{R}_{G,T_\varphi}$  arising from an ergodic measure preserving action  $T_\varphi$  on a probability measure space  $(X, \mu)$ . It is clear that  $\mathcal{F}(\mathcal{R}_{G,\varphi}) \subseteq \mathcal{F}(\mathcal{R}_{G,T_\varphi})$ .

For an additive subgroup  $E$  of  $\mathbb{R}$  containing 1, we define the positive inner multiplier group  $IM_+(E)$  of  $E$  by

$$IM_+(E) = \{t \in \mathbb{R}_+^\times \mid t \in E, t^{-1} \in E, \text{ and } tE = E\}.$$

By Lemma 3.6 in [33], there exists a unital subring  $R$  of  $\mathbb{R}$  such that  $IM_+(E) = R_+^\times$ . Hence not all countable subgroups of  $\mathbb{R}_+^\times$  arise as  $IM_+(E)$ . For example,  $\{9^n \in \mathbb{R}_+^\times \mid n \in \mathbb{Z}\}$  does not arise as  $IM_+(E)$  for any additive subgroup  $E$  of  $\mathbb{R}$  containing 1.

**Proposition 4.4.** Let  $\mathcal{R}_{G,\varphi}$  be a uniquely ergodic Cantor minimal system with an  $\mathcal{R}_{G,\varphi}$ -invariant probability measure  $\mu$ . Then

$$\mathcal{F}(\mathcal{R}_{G,\varphi}) \subseteq IM_+(T_\mu(D_m(\mathcal{R}_{G,\varphi}))) = IM_+(\{\int_X f d\mu : [f] \in D_m(\mathcal{R}_{G,\varphi})\}).$$

*Proof.* Let  $\lambda \in \mathcal{F}(\mathcal{R}_{G,\varphi})$ . For any clopen set  $U$  in  $X \times \{1, \dots, n\}$ , there exist clopen sets  $V_1, \dots, V_n$  in  $X$  such that  $U = \cup_{k=1}^n V_k \times \{k\}$ . Hence Theorem 4.1 implies that  $\lambda, \lambda^{-1} \in T_\mu(D_m(\mathcal{R}_{G,\varphi}))$ . By the proof of Theorem 4.1, there exist homeomorphisms  $F_1$  and  $F_2$  of  $X \times \mathbb{N}$  such that  $\mu \times \delta \circ F_1 = \lambda\mu \times \delta$  and  $\mu \times \delta \circ F_2 = \lambda^{-1}\mu \times \delta$ . Let  $V$  be a clopen set in  $X$ . Then  $\mu \times \delta(F_1(V \times \{1\})) = \lambda\mu(V)$  and  $\mu \times \delta(F_2(V \times \{1\})) = \lambda^{-1}\mu(V)$ . Therefore we see that  $\lambda T_\mu(D_m(\mathcal{R}_{G,\varphi})) \subseteq T_\mu(D_m(\mathcal{R}_{G,\varphi}))$  and  $\lambda^{-1} T_\mu(D_m(\mathcal{R}_{G,\varphi})) \subseteq T_\mu(D_m(\mathcal{R}_{G,\varphi}))$  because  $F_1(V \times \{1\})$  and  $F_2(V \times \{1\})$  are compact. Consequently  $\lambda \in IM_+(T_\mu(D_m(\mathcal{R}_{G,\varphi})))$ .  $\square$

We shall show some examples.

**Example 4.5.** Let  $p$  be a prime number, and let  $X = \{0, 1, \dots, p-1\}^\mathbb{N}$ . Define  $\varphi$  by the addition of  $(1, 0, 0, \dots)$  with carry over the right and  $\varphi((p-1, p-1, \dots)) = (0, 0, \dots)$ . Then  $\mathcal{R}_{\mathbb{Z},\varphi}$  is a uniquely ergodic Cantor minimal system with  $T_\mu(D_m(\mathcal{R})) = \mathbb{Z}[\frac{1}{p}]$ . Put  $U = \{(x_n)_n \in X : x_1 = 0\}$ . It is easy to see that  $\mathcal{R}_{\mathbb{Z},\varphi}|_U$  is orbit equivalent to  $\mathcal{R}_{\mathbb{Z},\varphi}$  and  $\mu(U) = \frac{1}{p}$ . (In particular, the induced homeomorphism  $\varphi_U$  is topologically conjugate to  $\varphi$ .) Hence

$$\{p^n : n \in \mathbb{Z}\} \subseteq \mathcal{F}(\mathcal{R}_{\mathbb{Z},\varphi}) \subseteq IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z},\varphi}))) = \{p^n : n \in \mathbb{Z}\}.$$

Therefore  $\mathcal{F}(\mathcal{R}_{\mathbb{Z},\varphi}) = \{p^n : n \in \mathbb{Z}\}$ .

**Example 4.6.** Let  $p$  and  $q$  be prime numbers such that  $\gcd(p, q) = 1$ , and let  $X = \{0, \dots, p-1\}^\mathbb{N} \times \{0, \dots, q-1\}^\mathbb{N}$ . Define  $\varphi_1$  (resp.  $\varphi_2$ ) by  $\varphi_1((x, y)) = (\tilde{\varphi}_1(x), y)$  (resp.  $\varphi_2((x, y)) = (x, \tilde{\varphi}_2(y))$ ) for  $(x, y) \in \{0, \dots, p-1\}^\mathbb{N} \times \{0, \dots, q-1\}^\mathbb{N}$  where  $\tilde{\varphi}_1$  and  $\tilde{\varphi}_2$  are the homeomorphisms in Example 4.5. Then  $\mathcal{R}_{\mathbb{Z}^2, (\varphi_1, \varphi_2)}$  is a uniquely ergodic Cantor minimal system with  $T(D_m(\mathcal{R})) = \mathbb{Z}[\frac{1}{p}, \frac{1}{q}]$ . Put  $V = \{(x_n)_n \in$

$\{0, \dots, p-1\}^{\mathbb{N}} : x_1 = 0\}$  and  $U = V \times \{0, \dots, q-1\}^{\mathbb{N}}$ . Then  $\mathcal{R}_{\mathbb{Z}^2, (\varphi_1, \varphi_2)}|_U$  is equal to  $\mathcal{R}_{\mathbb{Z}^2, (\varphi_{1U}, \varphi_2)}$  where  $\varphi_{1U}$  is the induced homeomorphism of  $\varphi_1$  on  $U$ . Hence it is easy to see that  $\mathcal{R}_{\mathbb{Z}^2, (\varphi_1, \varphi_2)}|_U$  is orbit equivalent to  $\mathcal{R}_{\mathbb{Z}^2, (\varphi_1, \varphi_2)}$ . Therefore we see that  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}^2, (\varphi_1, \varphi_2)}) = \{p^n q^m : n, m \in \mathbb{Z}\}$ . by a similar argument as in Example 4.5.

**Example 4.7.** If  $\mathcal{R}_{\mathbb{F}^n, \varphi}$  is a uniquely ergodic Cantor minimal system arising from a free action of a non-amenable free group of finite rank, then  $\mathcal{F}(\mathcal{R}_{\mathbb{F}^n, \varphi}) = \{1\}$  by Remark 4.3 (iii) and [13]. Note that we do not know whether there exists such a example.

We shall show more interesting examples in the next section. Giordano-Matui-Putnam-Skau classification theorem enable us to compute many examples.

**Theorem 4.8.** Let  $\mathcal{R}_{G, \varphi}$  be a uniquely ergodic Cantor minimal system arising from a free action of a finitely generated abelian group. Then

$$\mathcal{F}(\mathcal{R}_{G, \varphi}) = IM_+(T_\mu(D_m(\mathcal{R}))) = IM_+(\left\{ \int_X f d\mu : [f] \in D_m(\mathcal{R}_{G, \varphi}) \right\}).$$

*Proof.* By the result in [16],  $\mathcal{R}_{G, \varphi}$  is orbit equivalent to a Cantor minimal  $\mathbb{Z}$ -system. Hence we may assume that  $G = \mathbb{Z}$ .

We have  $\mathcal{F}(\mathcal{R}_{G, \varphi}) \subseteq IM_+(T_\mu(D_m(\mathcal{R}_{G, \varphi})))$  by Proposition 4.4. Conversely, let  $\lambda$  be an element in  $IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi})))$  with  $\lambda \leq 1$ . Then we see that there exists a clopen set  $U$  in  $X$  such that  $\mu(U) = \lambda$  by Lemma 2.5 in [20]. Define an additive homomorphism  $\psi$  of  $T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi}))$  by  $\psi(t) = \lambda t$ . Then  $\psi$  is an order isomorphism and  $\psi(1) = \lambda$ . Proposition 2.3 implies that

$$(D_m(\mathcal{R}_{\mathbb{Z}, \varphi}|_U), D_m(\mathcal{R}_{\mathbb{Z}, \varphi}|_U)_+, [1_U]) \cong (T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi})), T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi}))_+, \lambda).$$

Hence  $\mathcal{R}_{\mathbb{Z}, \varphi}|_U$  is orbit equivalent to  $\mathcal{R}_{\mathbb{Z}, \varphi}$  by Theorem 2.2 in [17] because  $\mathcal{R}_{\mathbb{Z}, \varphi}|_U$  is a Cantor minimal  $\mathbb{Z}$ -system (see Section 2 and [17]). Therefore  $\lambda \in \mathcal{F}(\mathcal{R}_{G, \varphi})$ .  $\square$

**Corollary 4.9.** If  $R$  is a countable unital subring of  $\mathbb{R}$ , then there exists a uniquely ergodic Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi}) = R_+^\times$ . Conversely if  $\mathcal{R}_{G, \varphi}$  arises from a uniquely ergodic free action of a finitely generated abelian group, then there exists a countable unital subring  $R$  of  $\mathbb{R}$  such that  $\mathcal{F}(\mathcal{R}_{G, \varphi}) = R_+^\times$ .

*Proof.* Let  $R$  be a countable unital subring of  $\mathbb{R}_+^\times$ . By Corollary 6.3 in [21], there exists a uniquely ergodic Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that

$$(D_m(\mathcal{R}_{\mathbb{Z}, \varphi}), D_m(\mathcal{R}_{\mathbb{Z}, \varphi})_+, [1_X]) \cong (R, R_+, 1).$$

Therefore we see that  $\mathcal{F}(\mathcal{R}_{G, \varphi}) = R_+^\times$  by Theorem 4.8.

Conversely, let  $\mathcal{R}_{G, \varphi}$  be a uniquely ergodic Cantor minimal system arising from a free action of a finitely generated abelian group. Then we have  $\mathcal{F}(\mathcal{R}_{G, \varphi}) = IM_+(T_\mu(D_m(\mathcal{R})))$  by Theorem 4.8. Therefore Lemma 3.6 in [33] implies that there exists a countable unital subring  $R$  of  $\mathbb{R}$  such that  $\mathcal{F}(\mathcal{R}_{G, \varphi}) = R_+^\times$ .  $\square$

By the theorem above,  $\{9^n : n \in \mathbb{Z}\}$  cannot be realized as the fundamental group of a uniquely ergodic Cantor minimal system arising from a free action of a finitely generated abelian group.

Let  $A$  be a simple  $C^*$ -algebra with a unique trace  $\tau$ . The fundamental group  $\mathcal{F}(A)$  of  $A$  is defined by the set

$$\{\tau \otimes Tr(p) \in \mathbb{R}_+^\times \mid p \text{ is a projection in } M_n(A) \text{ such that } pM_n(A)p \cong A\}$$

where  $Tr$  is the unnormalized trace on  $M_n(\mathbb{C})$ . We refer the reader to [3] for basic facts of operator algebras. We denote by  $\tau_*$  the map  $K_0(A) \rightarrow \mathbb{R}$  induced by a trace  $\tau$  on  $A$ . By Proposition 3.7 in [33], we have  $\mathcal{F}(A) \subseteq IM_+(\tau_*(K_0(A)))$ . If  $\varphi$  is a uniquely ergodic free minimal action of  $G$  on  $X$ , then the associated  $C^*$ -algebra

$C(X) \rtimes_{\varphi} G$  is a simple  $C^*$ -algebra with unique trace. We do not know whether there exists a relation between orbit equivalence of Cantor minimal  $G$ -systems and  $C^*$ -isomorphism of associated  $C^*$ -algebras in general. But we have the following theorem.

**Theorem 4.10.** Let  $\mathcal{R}_{\mathbb{Z}^n, \varphi}$  be a uniquely ergodic Cantor minimal system arising from a free action of a finitely generated free abelian group  $\mathbb{Z}^n$ . Then  $\mathcal{F}(C(X) \rtimes_{\varphi} \mathbb{Z}^n) \subseteq \mathcal{F}(\mathcal{R}_{\mathbb{Z}^n, \varphi})$ .

*Proof.* The Gap Labeling Theorem (Theorem D.1 in [30], see also [1], [2] and [22]) implies that

$$T_{\mu}(D_m(\mathcal{R}_{\mathbb{Z}^n, \varphi})) = \tau_*(K_0(C(X) \rtimes_{\varphi} \mathbb{Z}^n)).$$

By Theorem 4.8,  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}^n, \varphi}) = IM_+(T(D_m(\mathcal{R}_{\mathbb{Z}^n, \varphi})))$ . Hence we see that  $\mathcal{F}(C(X) \rtimes_{\varphi} \mathbb{Z}) \subseteq \mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi})$  because we have  $\mathcal{F}(C(X) \rtimes_{\varphi} \mathbb{Z}) \subseteq IM_+(\tau_*(K_0(C(X) \rtimes_{\varphi} \mathbb{Z})))$  by Proposition 3.7 in [33].  $\square$

Note that we do not know whether  $C(X) \rtimes_{\varphi} \mathbb{Z}^2$  belongs to classifiable classes by  $K$ -groups. We shall show that there exists a uniquely ergodic Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that  $\mathcal{F}(C(X) \rtimes_{\varphi} \mathbb{Z}) \neq \mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi})$ . By Corollary 6.3 in [21] and [9], there exists a uniquely ergodic Cantor minimal  $\mathbb{Z}$ -system  $\mathcal{R}_{\mathbb{Z}, \varphi}$  such that

$$K_0(C(X) \rtimes_{\varphi} \mathbb{Z}) = \{(\frac{b}{9^a}, c) \in \mathbb{R} \times \mathbb{Z} \mid a, b, c \in \mathbb{Z}, b \equiv c \pmod{8}\},$$

$$K_0(C(X) \rtimes_{\varphi} \mathbb{Z})_+ = \{(\frac{b}{9^a}, c) \in K_0(A) : \frac{b}{9^a} > 0\} \cup \{(0, 0)\} \text{ and } [1_A]_0 = (1, 1).$$

Then we see that  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi}) = \{3^n : n \in \mathbb{Z}\}$  and  $\mathcal{F}(C(X) \rtimes_{\varphi} \mathbb{Z}) = \{9^n : n \in \mathbb{Z}\}$ . See Example 3.12 in [34].

## 5. EXAMPLES

First we shall consider a generalization of Example 4.5. Let  $\{n_i\}_{i \in \mathbb{N}}$  be a sequence of natural numbers, each greater than or equal to 2 and let  $X = \prod_{i \in \mathbb{N}} \{0, \dots, n_i - 1\}$ . Define a homeomorphism  $\varphi$  by the addition of  $(1, 0, 0, \dots)$  with carry over the right and  $\varphi((n_1 - 1, n_2 - 1, \dots)) = (0, 0, \dots)$ . Then  $\mathcal{R}_{\mathbb{Z}, \varphi}$  is a uniquely ergodic Cantor minimal system. This system is called the *odometer system* associated with  $\{n_i\}_{i \in \mathbb{N}}$ . For each prime number  $p$ , define  $\epsilon_p$  in  $\mathbb{Z}_+ \cup \{\infty\}$  by  $\sup\{m : p^m \text{ divides } \prod_{i=1}^k n_i \text{ for some } k \in \mathbb{N}\}$ . Let  $S$  be the set of prime numbers and  $N := \prod_{p \in S} p^{\epsilon_p}$ . We call  $N$  the *super natural number*. It is known that two odometer systems  $\varphi_1$  and  $\varphi_2$  are topologically conjugate if and only if their super natural numbers are equal.

**Proposition 5.1.** Let  $\mathcal{R}_{\mathbb{Z}, \varphi}$  be an odometer system associated with  $\{n_i\}_{i \in \mathbb{N}}$ . Then

$$\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi}) = \{p_1^{m_1} \cdots p_n^{m_n} : n \in \mathbb{N}, \epsilon_{p_i} = \infty, m_i \in \mathbb{N}\}$$

where  $\epsilon(p) = \sup\{m : p^m \text{ divides } \prod_{i=1}^k n_i \text{ for some } k \in \mathbb{N}\}$ .

*Proof.* Since  $T_{\mu}(D_m(\mathcal{R}_{\mathbb{Z}, \varphi})) = \{\frac{n}{q} : n \in \mathbb{Z}, q = p^m \text{ where } p \text{ is a prime number and } 0 \leq m \leq \epsilon_p, m < \infty\}$ , Theorem 4.8 implies the conclusion.  $\square$

We shall consider fundamental groups of certain Denjoy systems. A *Denjoy homeomorphism*  $\varphi$  is an homeomorphism of  $\mathbb{T}$  such that the rotation number  $\rho(\varphi)$  is irrational and  $\varphi$  is not conjugate to a pure rotation. Such a homeomorphism has a unique minimal set  $\Sigma$  which is homeomorphic to the Cantor set. The restriction of a Denjoy homeomorphism to its unique minimal set  $\Sigma$  is called the *Denjoy system*. It is known that a Denjoy system  $\mathcal{R}_{\mathbb{Z}, \varphi|_{\Sigma}}$  is a uniquely ergodic Cantor minimal system. We refer the reader to [27], [28] and [39] for details of Denjoy systems. It is known that for a Denjoy homeomorphism  $\varphi$  there exists an orientation preserving



surjective continuous map  $h$  of  $\mathbb{T}$  to  $\mathbb{T}$  such that  $h \circ \varphi = R_{\rho(\varphi)} \circ h$  where  $R_{\rho(\varphi)}$  is the rotation such that  $R_{\rho(\varphi)}([t]) = [t + \rho(\varphi)]$ . (We regard  $\mathbb{T}$  as  $\{e^{2\pi i t} : t \in \mathbb{R}\}$ .) With notation as above, put  $Q(\varphi) = h(\{z \in \mathbb{Z} : \sharp(h^{-1}(h(z))) \neq 1\})$ . The set  $Q(\varphi)$  is countable and  $R_{\rho(\varphi)}$ -invariant. Conversely, any countable  $R_{\rho(\varphi)}$ -invariant subset of  $\mathbb{T}$  can be realized as  $Q(\varphi)$  of some Denjoy homeomorphism  $\varphi$ . Markley showed that two Denjoy homeomorphisms  $\varphi_1$  and  $\varphi_2$  are conjugate via an orientation preserving conjugating map if and only if  $\rho(\varphi_1) = \rho(\varphi_2)$  and  $Q(\varphi_1) = R_\theta(Q(\varphi_2))$  for some  $\theta \in (0, 1]$ .

**Proposition 5.2.** Let  $\theta$  be an irrational number in  $[0, 1]$  and  $\varphi_\theta$  a Denjoy homeomorphism with  $\rho(\varphi_\theta) = \theta$  and  $Q(\varphi_\theta) = \{e^{2\pi i n \theta} : n \in \mathbb{Z}\}$ . Then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma})$  is a singly generated group. More precisely, if  $\theta$  is not a quadratic number, then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = \{1\}$ . If  $\theta$  is a quadratic number, then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = \{\epsilon_0^n \in R_+^\times \mid n \in \mathbb{Z}\}$  and the generator  $\epsilon_0$  is given by  $\epsilon_0 = \frac{t+u\sqrt{D_\theta}}{2}$ , where  $D_\theta$  is the discriminant of  $\theta$  and  $t, u$  are the positive integers satisfying one of the Pell equations  $t^2 - D_\theta u^2 = \pm 4$  and  $\frac{t+u\sqrt{D_\theta}}{2}$  exceeds 1 and is least.

*Proof.* By theorem 4.8, we see that  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma})))$ . Since we have  $T_\mu(D_m(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma})) = \mathbb{Z} + \mathbb{Z}\theta$  (see, for example, [39]), the same argument as in [33](Corollary 3.18) implies the conclusion.  $\square$

We shall show some examples.

**Example 5.3.** Let  $\theta = \frac{-1+\sqrt{5}}{2}$ . Then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = \{(\frac{1+\sqrt{5}}{2})^n : n \in \mathbb{Z}\}$ .

**Example 5.4.** Let  $\theta = \sqrt{5} - 2$ . Then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = \{(\sqrt{5} - 2)^n : n \in \mathbb{Z}\}$ .

**Example 5.5.** Let  $\theta = \frac{1}{\sqrt{5}}$ . Then  $\mathcal{F}(\mathcal{R}_{\mathbb{Z}, \varphi_\theta|_\Sigma}) = \{(\sqrt{5} - 2)^n : n \in \mathbb{Z}\}$ .

We shall consider a natural generalization to  $\mathbb{Z}^2$ -actions of Denjoy systems. Let  $\theta_1$  and  $\theta_2$  be irrational numbers and  $C$  an countable invariant subset of  $\mathbb{T}$  under the rotations  $R_{\theta_1}$  and  $R_{\theta_2}$ . We can construct a Cantor set  $X$  by disconnecting the circle  $\mathbb{T}$  along any point of  $C$ . Moreover there exist commuting homeomorphisms  $\varphi_1$  and  $\varphi_2$  of  $X$ , which are extensions of  $R_{\theta_1}$  and  $R_{\theta_2}$  respectively. See chapter 3 in [8], [12], Example 4.4 in [19] and [39] for details and precise definition. It is known that this Cantor  $\mathbb{Z}^2$ -system is a uniquely ergodic Cantor minimal system. Moreover the invariant probability measure of this Cantor minimal system comes from the normalized Haar measure on  $\mathbb{T}$ . We call this Cantor system the *Denjoy  $\mathbb{Z}^2$ -system* associated with  $(\theta_1, \theta_2, C)$ . We need algebraic number theory of cubic fields to compute some examples of fundamental groups of Denjoy  $\mathbb{Z}^2$ -systems. We refer the reader to [41] for algebraic number theory of cubic fields.

**Example 5.6.** Let  $\mathcal{R}_{\mathbb{Z}^2, \varphi}$  be a Denjoy  $\mathbb{Z}^2$ -system associated with  $\theta_1 = \sqrt[3]{2}$ ,  $\theta_2 = \sqrt[3]{4}$  and  $C = \{e^{2\pi i(n\sqrt[3]{2}+m\sqrt[3]{4})} : n, m \in \mathbb{Z}\}$ . Then we have  $T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})) = \mathbb{Z} + \mathbb{Z}\sqrt[3]{2} + \mathbb{Z}\sqrt[3]{4}$ . Since  $\mathbb{Z} + \mathbb{Z}\sqrt[3]{2} + \mathbb{Z}\sqrt[3]{4}$  is the ring of integers on a cubic field  $\mathbb{Q}(\sqrt[3]{2})$ ,  $IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})))$  is the positive part of the unit group of  $\mathbb{Q}(\sqrt[3]{2})$ , which is equal to  $\{(1 + \sqrt[3]{2} + \sqrt[3]{4})^n : n \in \mathbb{Z}\}$ . (See Theorem 13.6.2 in [41].) Therefore Theorem 4.8 implies

$$\mathcal{F}(\mathcal{R}_{\mathbb{Z}^2, \varphi}) = \{(1 + \sqrt[3]{2} + \sqrt[3]{4})^n : n \in \mathbb{Z}\}.$$

**Example 5.7.** Let  $\mathcal{R}_{\mathbb{Z}^2, \varphi}$  be a Denjoy  $\mathbb{Z}^2$ -system associated with  $\theta_1 = \sqrt[3]{3}$ ,  $\theta_2 = \sqrt[3]{9}$  and  $C = \{e^{2\pi i(n\sqrt[3]{3}+m\sqrt[3]{9})} : n, m \in \mathbb{Z}\}$ . Then we have  $T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})) = \mathbb{Z} + \mathbb{Z}\sqrt[3]{3} + \mathbb{Z}\sqrt[3]{9}$ . Since  $\mathbb{Z} + \mathbb{Z}\sqrt[3]{3} + \mathbb{Z}\sqrt[3]{9}$  is the ring of integers on a cubic field  $\mathbb{Q}(\sqrt[3]{3})$ ,  $IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})))$  is the positive part of the unit group of  $\mathbb{Q}(\sqrt[3]{3})$ , which is

equal to  $\{(4 + 3\sqrt[3]{2} + 2\sqrt[3]{4})^n : n \in \mathbb{Z}\}$ . (See [41] or Appendix B.3 in [5].) Therefore Theorem 4.8 implies

$$\mathcal{F}(\mathcal{R}_{\mathbb{Z}^2, \varphi}) = \{(4 + 3\sqrt[3]{2} + 2\sqrt[3]{4})^n : n \in \mathbb{Z}\}.$$

**Example 5.8.** Let  $\mathcal{R}_{\mathbb{Z}^2, \varphi}$  be a Denjoy  $\mathbb{Z}^2$ -system associated with  $\theta_1 = 2 \cos \frac{2\pi}{7}$ ,  $\theta_2 = 4 \cos^2 \frac{2\pi}{7}$  and  $C = \{e^{2\pi i(n2 \cos \frac{2\pi}{7} + m4 \cos^2 \frac{2\pi}{7})} : n, m \in \mathbb{Z}\}$ . Then we have  $T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})) = \mathbb{Z} + \mathbb{Z}2 \cos \frac{2\pi}{7} + \mathbb{Z}4 \cos^2 \frac{2\pi}{7}$ . Note that  $2 \cos \frac{2\pi}{7}$  is a root of the equation  $x^3 + x^2 - 2x - 1 = 0$ . Since  $\mathbb{Z} + \mathbb{Z}2 \cos \frac{2\pi}{7} + \mathbb{Z}4 \cos^2 \frac{2\pi}{7}$  is the ring of integers on a cubic field  $\mathbb{Q}(2 \cos \frac{2\pi}{7})$ ,  $IM_+(T_\mu(D_m(\mathcal{R}_{\mathbb{Z}^2, \varphi})))$  is the positive part of the unit group of  $\mathbb{Q}(2 \cos \frac{2\pi}{7})$ , which is equal to  $\{(-1 + 2 \cos \frac{2\pi}{7} + 4 \cos^2 \frac{2\pi}{7})^n (2 - 4 \cos^2 \frac{2\pi}{7})^m : n, m \in \mathbb{Z}\}$ . (See Appendix B.4 in [5].) Therefore Theorem 4.8 implies

$$\mathcal{F}(\mathcal{R}_{\mathbb{Z}^2, \varphi}) = \{(-1 + 2 \cos \frac{2\pi}{7} + 4 \cos^2 \frac{2\pi}{7})^n (2 - 4 \cos^2 \frac{2\pi}{7})^m : n, m \in \mathbb{Z}\}.$$

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